

Average best m -term approximation

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Abstract

We introduce the concept of average best m -term approximation widths with respect to a probability measure on the unit ball or the unit sphere of ℓ_p^n . We estimate these quantities for the embedding $id : \ell_p^n \rightarrow \ell_q^n$ with $0 < p \leq q \leq \infty$ for the normalized cone and surface measure. Furthermore, we consider certain tensor product weights and show that a typical vector with respect to such a measure exhibits a strong compressible (i.e. nearly sparse) structure. This measure may be therefore used as a random model for sparse signals.

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1 Introduction

1.1 Best m -term approximation

Let $m \in \mathbb{N}_0$ and let Σ_m be the set of all sequences $x = \{x_j\}_{j=1}^\infty$ with

$$\|x\|_0 := \#\text{supp } x = \#\{n \in \mathbb{N} : x_n \neq 0\} \leq m.$$

Here stands $\#A$ for the number of elements of a set A . The elements of Σ_m are said to be m -sparse. Observe, that Σ_m is a non-linear subset of every $\ell_q := \{x = \{x_j\}_{j=1}^\infty : \|x\|_q < \infty\}$, where

$$\|x\|_q := \begin{cases} \left(\sum_{j=1}^\infty |x_j|^q \right)^{1/q}, & 0 < q < \infty, \\ \sup_{j \in \mathbb{N}} |x_j|, & q = \infty. \end{cases}$$

For every $x \in \ell_q$, we define its *best m -term approximation error* by

$$\sigma_m(x)_q := \inf_{y \in \Sigma_m} \|x - y\|_q.$$

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Moreover for $0 < p \leq q \leq \infty$, we introduce the *best m -term approximation widths*

$$\sigma_m^{p,q} := \sup_{x: \|x\|_p \leq 1} \sigma_m(x)_q.$$

The use of this concept goes back to Schmidt [44] and after the work of Oskolkov [39], it was widely used in the approximation theory, cf. [15, 18, 45]. In fact, it is the main prototype of nonlinear approximation [17]. It is well known, that

$$2^{-1/p}(m+1)^{1/q-1/p} \leq \sigma_m^{p,q} \leq (m+1)^{1/q-1/p}, \quad m = 0, 1, 2, \dots \quad (1)$$

The proof of (1) is based on the simple fact, that (roughly speaking) the best m -term approximation error of $x \in \ell_p$ is realized by subtracting the m largest coefficients taken in absolute value. Hence,

$$\sigma_m(x)_q = \begin{cases} \left(\sum_{j=m+1}^{\infty} (x_j^*)^q \right)^{1/q}, & 0 < q < \infty, \\ x_{m+1}^* = \sup_{j \geq m+1} x_j^*, & q = \infty, \end{cases}$$

where $x^* = (x_1^*, x_2^*, \dots)$ denotes the so-called *non-increasing rearrangement* [6] of the vector $(|x_1|, |x_2|, |x_3|, \dots)$.

Let us recall the proof of (1) in the simplest case, namely $q = \infty$. The estimate from above then follows by

$$\sigma_m(x)_{\infty} = \sup_{j \geq m+1} x_j^* = x_{m+1}^* \leq \left((m+1)^{-1} \sum_{j=1}^{m+1} (x_j^*)^p \right)^{1/p} \leq (m+1)^{-1/p} \|x\|_p. \quad (2)$$

The lower estimate is supplied by taking

$$x = (m+1)^{-1/p} \sum_{j=1}^{m+1} e_j, \quad (3)$$

where $\{e_j\}_{j=1}^{\infty}$ are the canonical unit vectors.

For general q , the estimate from above in (1) may be obtained from (2) and Hölder's inequality

$$\|x\|_q \leq \|x\|_p^{\theta} \cdot \|x\|_{\infty}^{1-\theta}, \quad \text{where} \quad \frac{1}{q} = \frac{\theta}{p}. \quad (4)$$

The estimate from below follows for all q 's by simple modification of (3).

The discussion above exhibits two effects.

- (i) Best m -term approximation works particularly well, when $1/p - 1/q$ is large, i.e. if $p < 1$ and $q = \infty$.
- (ii) The elements used in the estimate from below (and hence the elements, where the best m -term approximation performs at worse) enjoy a very special structure.

Therefore, there is a reasonable hope, that the best m -term approximation could behave better, when considered in a certain average case. But first we point out two different interesting points of view on the subject.

1.2 Connection to compressed sensing

The interest in ℓ_p spaces (and especially in their finite-dimensional counterparts ℓ_p^n) with $0 < p < 1$ was recently stimulated by the impressive success of the novel and vastly growing area of *compressed sensing* as introduced in [8, 10, 11, 19]. Without going much into the details, we only note, that the techniques of compressed sensing allow to reconstruct a vector from an incomplete set of measurements utilizing the prior knowledge, that it is sparse, i.e. $\|x\|_0$ is small. Furthermore, this approach may be applied [14] also to vectors, which are *compressible*, i.e. $\|x\|_p$ is small for (preferably small) $0 < p < 1$. Indeed, (1) tells us, that such a vector x may be very well approximated by sparse vectors. We point to [9, 24, 25, 42] for the current state of the art of this field and for further references.

This leads in a very natural way to a question, which stands in the background of this paper, namely:

How does a typical vector of the ℓ_p^n unit ball look like?

or, posed in an exact way:

Let μ be a probability measure on the unit ball of ℓ_p^n . What is the mean value of $\sigma_m(x)_q$ with respect to this measure?

Of course, the choice of μ plays a crucial role. There are several standard probability measures, which are connected to the unit ball of ℓ_p^n in a natural way, namely (cf. Definitions 2 and 9)

- (i) the normalized Lebesgue measure,
- (ii) the $n - 1$ dimensional Hausdorff measure restricted to the surface of the unit ball of ℓ_p^n and correspondingly normalized,
- (iii) the so-called normalized cone measure.

Unfortunately, it turns out, that all these three measures are “bad” – a typical vector with respect to any of them does not involve much structure and corresponds rather to noise than signal (in the sense described below). Therefore, we are looking for a new type of measures (cf. Definition 13), which would behave better from this point of view.

1.3 Random models of noise and signals

Random vectors play an important role in the area of signal processing. For example, if $n \in \mathbb{N}$ is a natural number, $\omega = (\omega_1, \dots, \omega_n)$ is a vector of independent Gaussian variables and $\varepsilon > 0$ is a real number, then $\varepsilon\omega$ is a classical model of noise, namely the *white noise*. This model is used in the theory but also in the real life applications of signal processing.

The random generation of a structured signal seems to be a more complicated task. Probably the most common random model to generate sparse vectors, cf. [7, 13, 30, 40], is the so-called *Bernoulli-Gaussian model*. Let again $n \in \mathbb{N}$ be a

natural number and $\varepsilon > 0$ be a real number. Also $\omega = (\omega_1, \dots, \omega_n)$ stands for a vector of independent Gaussian variables. Furthermore, let $0 < p < 1$ be a real number and let $\varrho = (\varrho_1, \dots, \varrho_n)$ be a vector of independent Bernoulli variables defined as

$$\varrho_i = \begin{cases} 1, & \text{with probability } p, \\ 0, & \text{with probability } 1 - p. \end{cases}$$

The components of the random *Bernoulli-Gaussian vector* $x = (x_1, \dots, x_n)$ are then defined through

$$x_i = \varepsilon \varrho_i \cdot \omega_i, \quad i = 1, \dots, n. \quad (5)$$

Obviously, the average number of non-zero components of x is $k := pn$. Unfortunately, if k is much smaller than n , then the concentration of the number of non-zero components of x around k is not very strong. This becomes better, if k gets larger. But in that case, the model (5) resembles more and more the model of white noise. In some sense, (5) represents rather a randomly filtered white noise than a structured signal. It is one of the main aims of this paper to find a new measure, such that a random vector with respect to this measure would show a nearly sparse structure without the need of random filtering.

1.4 Unit sphere

Let us describe the situation in the most prominent case, when $p = 2$, $m = 0$ and $\mu = \mu_2$ is the normalized surface measure on the unit sphere \mathbb{S}^{n-1} of ℓ_2^n . Furthermore, we denote by γ_n the standard Gaussian measure on \mathbb{R}^n with the density

$$\frac{1}{(2\pi)^{n/2}} e^{-\|x\|_2^2/2}, \quad x \in \mathbb{R}^n.$$

We use polar coordinates to calculate

$$\begin{aligned} \int_{\mathbb{R}^n} \max_{j=1, \dots, n} |x_j| d\gamma_n(x) &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \max_{j=1, \dots, n} |x_j| \cdot e^{-\|x\|_2^2/2} dx \\ &= \frac{\Omega_n}{(2\pi)^{n/2}} \int_0^\infty r^{n-1} \int_{\mathbb{S}^{n-1}} \max_{j=1, \dots, n} |rx_j| e^{-\|rx\|_2^2/2} d\mu_2(x) dr \\ &= \frac{\Omega_n}{(2\pi)^{n/2}} \int_0^\infty r^n e^{-r^2/2} dr \cdot \int_{\mathbb{S}^{n-1}} \max_{j=1, \dots, n} |x_j| d\mu_2(x) \quad (6) \\ &= \frac{\Omega_n}{(2\pi)^{n/2}} \int_0^\infty r^n e^{-r^2/2} dr \cdot \int_{\mathbb{S}^{n-1}} \sigma_0(x)_\infty d\mu_2(x), \end{aligned}$$

where Ω_n denotes the area of \mathbb{S}^{n-1} . This formula connects the expected value of $\sigma_0(x)_\infty$ with the expected value of maximum of n independent Gaussian variables. Using that this quantity is known to be equivalent to $\sqrt{\log(n+1)}$, cf. [33, (3.14)],

$$\int_0^\infty r^n e^{-r^2/2} dr = 2^{(n-1)/2} \Gamma((n+1)/2) \quad \text{and} \quad \Omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)},$$

one obtains

$$\int_{\mathbb{S}^{n-1}} \sigma_0(x)_\infty d\mu_2(x) \approx \sqrt{\frac{\log(n+1)}{n}}, \quad n \in \mathbb{N}. \quad (7)$$

Several comments on (6) and (7) are necessary.

- (i) Quantities similar to the left-hand side of (7) have been used in the study of geometry of Banach spaces and local theory of Banach spaces since many years and are treated in detail in the work of Milman [23, 35, 36]. Especially, if $\|\cdot\|_K$ is a norm in \mathbb{R}^n and $K := \{x \in \mathbb{R}^n : \|x\|_K \leq 1\}$ denotes the corresponding unit ball, then the quantity

$$A_K = \int_{\mathbb{S}^{n-1}} \|x\|_K d\mu_2(x)$$

(and the closely connected median M_K of $\|x\|_K$ over \mathbb{S}^{n-1}) plays a crucial role in the Dvoretzky theorem [20, 22, 35] and, in general, in the study of Euclidean sections of K , cf. [36, Section 5]. Furthermore, it is known that the case of $K = [-1, 1]^n$, when

$$A_K = \int_{\mathbb{S}^{n-1}} \max_{j=1, \dots, n} |x_j| d\mu_2(x) = \int_{\mathbb{S}^{n-1}} \sigma_0(x)_\infty d\mu_2(x),$$

is extremal, cf. [35].

- (ii) The connection between the estimated value of a maximum of independent Gaussian variables and the estimated value of the largest coordinate of a random vector on \mathbb{S}^{n-1} is given just by integration in polar coordinates and is one of the standard techniques in the local theory of Banach spaces. Due to the result of [43], this holds true also for other values of p , even for $p < 1$, with Gaussian variables replaced by variables with the density $c_p e^{-|t|^p}$. This approach is nowadays classical in the study of the geometry and concentration of measure phenomenon on the ℓ_p^n -balls, cf. [2, 3, 4, 5, 37, 38, 41].
- (iii) For every $x \in \mathbb{S}^{n-1}$ we obtain easily that $\max_{j=1, \dots, n} |x_j| \geq \left(\frac{1}{n} \sum_{j=1}^n x_j^2\right)^{1/2} = 1/\sqrt{n}$. Estimate (7) shows that the average value of $\max_{j=1, \dots, n} |x_j|$ over \mathbb{S}^{n-1} is asymptotically larger only by a logarithmic factor. The detailed study of the concentration of $\max_{j=1, \dots, n} |x_j|$ around its estimated value (or its mean value) is known as *concentration of measure phenomena* [32, 33, 36] and gives more accurate information than the one included in (7). As our main interest lies in estimates of *average best m -term widths*, cf. Definition 1, we do not investigate the concentration properties in this paper and leave this subject to further research.
- (iv) The calculation (6) is based on the use of polar coordinates. For $p \neq 2$, the normalized cone measure is exactly that measure, for which a similar formula holds, cf. (13). The estimates for $n - 1$ dimensional surface measure are later obtained using its density with respect to the cone measure, cf. Lemma 10.
- (v) As we want to keep the paper self-contained as much as possible and to make it readable also for readers without (almost) any stochastic background, we prefer to use simple and direct techniques. For example we use rather the simple estimates in Lemma 5, than any of their sophisticated improvements available in literature.

- (vi) The connection to random Gaussian variables explains, why a random point of \mathbb{S}^{n-1} is sometimes referred to as *white (or Gaussian) noise*. It is usually not associated with any reasonable (i.e. structured) signal, rather it represents a good model for random noise.

1.5 Basic Definitions and Main Results

1.5.1 Definition of average best m -term widths

After describing the context of our work we shall now present the definition of the so-called *average best m -term widths*, which are the main subject of our study.

First, we observe, that

$$\sigma_m((x_1, \dots, x_n))_q = \sigma_m((\varepsilon_1 x_1, \dots, \varepsilon_n x_n))_q = \sigma_m((|x_1|, \dots, |x_n|))_q$$

holds for every $x \in \mathbb{R}^n$ and $\varepsilon \in \{-1, +1\}^n$. Also all the measures, which we shall consider, are invariant under any of the mappings

$$(x_1, \dots, x_n) \rightarrow (\varepsilon_1 x_1, \dots, \varepsilon_n x_n), \quad \varepsilon \in \{-1, +1\}^n$$

and therefore we restrict our attention only to \mathbb{R}_+^n in the following definition.

Definition 1. Let $0 < p \leq q \leq \infty$ and let $n \geq 2$ and $0 \leq m \leq n - 1$ be natural numbers.

- (i) We set

$$\Delta_p^n = \begin{cases} \{(t_1, \dots, t_n) \in \mathbb{R}_+^n : \sum_{j=1}^n t_j^p = 1\}, & p < \infty, \\ \{(t_1, \dots, t_n) \in \mathbb{R}_+^n : \max_{j=1, \dots, n} t_j = 1\}, & p = \infty. \end{cases}$$

- (ii) Let μ be a Borel probability measure on Δ_p^n . Then

$$\sigma_m^{p,q}(\mu) = \int_{\Delta_p^n} \sigma_m(x)_q d\mu(x)$$

is called *average surface best m -term width of $\text{id} : \ell_p^n \rightarrow \ell_q^n$ with respect to μ* .

- (iii) Let ν be a Borel probability measure on $[0, 1] \cdot \Delta_p^n$. Then

$$\sigma_m^{p,q}(\nu) = \int_{[0,1] \cdot \Delta_p^n} \sigma_m(x)_q d\nu(x)$$

is called *average volume best m -term width of $\text{id} : \ell_p^n \rightarrow \ell_q^n$ with respect to ν* .

Let us observe, that the estimates

$$\sigma_m^{p,q}(\mu) \leq \sigma_m^{p,q} \quad \text{and} \quad \sigma_m^{p,q}(\nu) \leq \sigma_m^{p,q}$$

follow trivially by Definition 1. Furthermore, the mapping $x \rightarrow \sigma_m(x)_q$ is continuous and, therefore, measurable with respect to the Borel measure μ .

1.5.2 Main results

After introducing new notion of average best m -term width in Definition 1, we study its behavior for the measures on Δ_p^n , which are widely used in literature. A prominent role among them is played by the so-called *normalized cone measure* given by

$$\mu_p(\mathcal{A}) = \frac{\lambda([0, 1] \cdot \mathcal{A})}{\lambda([0, 1] \cdot \Delta_p^n)}, \quad \mathcal{A} \subset \Delta_p^n.$$

In Theorem 7 and Proposition 8 we provide basic estimates of $\sigma_m^{p,q}(\mu_p)$ for $q = \infty$ and $q < \infty$, respectively. Surprisingly enough, it turns out that (7) has its direct counterpart for all $0 < p < \infty$. This means (as described above), that the coordinates of a “typical” element of the surface of the ℓ_p^n unit ball are well concentrated around the value $n^{-1/p}$. So, roughly speaking, it is only ℓ_p -normalized noise.

Another well known probability measure on Δ_p^n is the *normalized surface measure* ϱ_p , cf. Definition 9. We calculate in Lemma 10 the density of ϱ_p with respect to μ_p to be equal to

$$\frac{d\varrho_p}{d\mu_p}(x) = c_{p,n}^{-1} \left(\sum_{i=1}^n x_i^{2p-2} \right)^{1/2},$$

where

$$c_{p,n} = \int_{\Delta_p^n} \left(\sum_{i=1}^n x_i^{2p-2} \right)^{1/2} d\mu_p(x)$$

is the normalizing constant. This result (which is a generalization of the work of Naor and Romik [38] to the non-convex case $0 < p < 1$) might be of independent interest for the study of the geometry of ℓ_p^n spheres. One observes immediately, that if $p < 1$ and one or more coordinates of x_i are going to zero, then this density has a polynomial singularity and, therefore, gives more weight to areas closed to coordinate hyperplanes.

We then obtain in Theorem 12 an estimate of $\sigma_0^{p,\infty}(\varrho_p)$ from above. Although the measure ϱ_p concentrates around coordinate hyperplanes, it turns out, that the estimate from above of $\sigma_0^{p,\infty}(\mu_p)$ as obtained in Theorem 7 and the estimate of Theorem 12 differ only in the constants involved.

The last part of this paper is devoted to the search of a new probability measure on Δ_p^n , which would “promote sparsity” in the sense, that the mean value of $\sigma_m(x)_q$ decays rapidly with m . One possible candidate is presented in Definition 13 by introducing a new class of measures $\theta_{p,\beta}$, which are given by their density with respect to the cone measure μ_p

$$\frac{d\theta_{p,\beta}}{d\mu_p}(x) = c_{p,\beta}^{-1} \cdot \prod_{i=1}^n x_i^\beta, \quad x \in \Delta_p^n,$$

where $c_{p,\beta}$ is a normalising constant. We refer also to Remark 4 for an equivalent characterisation.

We show, that for an appropriate choice of β , namely $\beta = p/n - 1$, the estimated value of the m -th largest coefficient of elements of the ℓ_p^n -unit sphere decays exponentially with m . Namely, Theorem 16 provides estimates of $\sigma_{m-1}^{p,\infty}(\theta_{p,p/n-1})$, which

at the end imply that

$$\frac{C_p^1}{\left(\frac{1}{p} + 1\right)^m} \leq \liminf_{n \rightarrow \infty} \sigma_{m-1}^{p,\infty}(\theta_{p,p/n-1}) \leq \limsup_{n \rightarrow \infty} \sigma_{m-1}^{p,\infty}(\theta_{p,p/n-1}) \leq \frac{C_p^2}{\left(\frac{1}{p} + 1\right)^m} \quad (8)$$

for two positive real numbers C_p^1 and C_p^2 , which depend only on p .

This result (which is also simulated numerically in the very last section of this paper) is in a certain way independent of n . This gives a hope, that one could apply this approach also to the infinite-dimensional spaces ℓ_p or, using a suitable discretization technique (like wavelet decomposition), also to some function spaces. This remains a subject of our further research.

Of course, the class $\theta_{p,\beta}$ provides only one example of measures with rapid decay of their average best m -term widths. We leave also the detailed study of other measures with such properties open to future work.

Note added in the proof: Let us comment on the relation of our work with recent papers of Cevher [12] and Gribonval, Cevher, and Davis [29]. Cevher uses in [12] the concept of *Order Statistics* [16] to identify the probability distributions, whose independent and identically distributed (i.i.d.) realizations result typically in p -compressible signals, i.e.

$$x_i^* \leq C R \cdot i^{-1/p}.$$

Our approach here is a bit different and more connected to the geometry of ℓ_p^n spaces. In accordance with [43], this leads to the study of ℓ_p^n -normalized vectors with i.i.d. components. This again allows us to better distinguish between the norm of such a vector (i.e. its *size* or *energy*) and its direction (i.e. its *structure*).

The approach of the recent preprint [29] (which was submitted during the review process of this work) comes much closer to ours. Their Definition 1 of “Compressible priors” introduces the quantity called *relative best m -term approximation error* as

$$\bar{\sigma}_m(x)_q = \frac{\sigma_m(x)_q}{\|x\|_q}, \quad x \in \mathbb{R}_+^n.$$

The asymptotic behavior of this quantity for $x = (x_1, \dots, x_n)$ being a vector with i.i.d. components and $\liminf_{n \rightarrow \infty} \frac{m_n}{n} \geq \kappa \in (0, 1)$ is then used to define q -compressible probability distribution functions. In contrary to [29], we consider ℓ_q approximation of ℓ_p normalized vectors and therefore our widths depend on two integrability parameters p and q . Furthermore, we do not pose any restrictions on the ratio m/n to any specific regime and consider the average best m -term widths $\sigma_m^{p,q}(\mu)$ for all $0 \leq m \leq n - 1$. In the only case, when we speak about asymptotics (i.e. (37) of Theorem 16), we suppose m to be constant and n growing to infinity. Furthermore, Theorem 1 of [29] shows that all distributions with bounded fourth moment do not fit into their scheme and do not “promote sparsity”. As we are interested in distributions, which are connected to the geometry of ℓ_p^n -balls (i.e. generalized Gaussian distribution and generalized Gamma distribution), it is exactly that reason why we change the parameters of the distribution $\theta_{p,\beta}$ in dependence of n . Although quite inconvenient from the mathematical point of view, it is not really clear if this presents a serious obstacle for application of our approach. But the investigation of this goes beyond the scope of this work.

1.5.3 Structure of the paper

The paper is structured as follows. The rest of Section 1 gives some notation used throughout the paper. Sections 2 and 3 provide estimates of this quantity with respect to the cone and surface measure, respectively. In Section 4, we study a new type of measures on the unit ball of ℓ_p^n . We show, that the typical element with respect to those measures behaves in a completely different way compared to the situations discussed before. Those results are illustrated by the numerical experiments described in Section 5.

1.6 Notation

We denote by \mathbb{R} the set of real numbers, by $\mathbb{R}_+ := [0, \infty)$ the set of nonnegative real numbers and by \mathbb{R}^n and \mathbb{R}_+^n their n -fold tensor products. The components of $x \in \mathbb{R}^n$ are denoted by x_1, \dots, x_n . The symbol λ stands for the Lebesgue measure on \mathbb{R}^n and \mathcal{H} for the $n - 1$ dimensional Hausdorff measure in \mathbb{R}^n . If $A \subset \mathbb{R}^n$ and $I \subset \mathbb{R}$ is an interval, we write $I \cdot A := \{tx : t \in I, x \in A\}$.

We shall use very often the *Gamma function*, defined by

$$\Gamma(s) := \int_0^\infty t^{s-1} e^{-t} dt, \quad s > 0. \quad (9)$$

In one case, we shall use also the *Beta function*

$$B(p, q) := \int_0^1 t^{p-1} (1-t)^{q-1} dt = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}, \quad p, q > 0 \quad (10)$$

and the *digamma function*

$$\Psi(s) := \frac{d}{ds} \log \Gamma(s) = \frac{\Gamma'(s)}{\Gamma(s)}, \quad s > 0.$$

We recommend [1, Chapter 6] as a standard reference for both basic and more advanced properties of these functions. We shall need the Stirling's approximation formula (which was implicitly used already in (7)) in its most simple form

$$\Gamma(x) = \sqrt{\frac{2\pi}{x}} \left(\frac{x}{e}\right)^x \left(1 + \mathcal{O}\left(\frac{1}{x}\right)\right), \quad x > 0. \quad (11)$$

If $a = \{a_j\}_{j=1}^\infty$ and $b = \{b_j\}_{j=1}^\infty$ are real sequences, then $a_j \lesssim b_j$ means, that there is an absolute constant $C > 0$, such that $a_j \leq C b_j$ for all $j = 1, 2, \dots$. Similar convention is used for $a_j \gtrsim b_j$ and $a_j \approx b_j$. The capital letter C with indices (i.e. C_p) denotes a positive real number depending only on the highlighted parameters and their meaning can change from one occurrence to another. If, for any reason, we shall need to distinguish between several numbers of this type, we shall write for example C_p^1 and C_p^2 as already done in (8).

2 Normalized cone measure

In this section, we study the average best m -term widths as introduced in Definition 1 for the most important measure (the so-called cone measure) on Δ_p^n , which is well studied in the literature within the geometry of ℓ_p^n spaces, cf. [38, 4, 37, 5]. Essentially, we recover in Theorem 7 an analogue of the estimate (7) for all $0 < p < \infty$.

Definition 2. Let $0 < p \leq \infty$ and $n \geq 2$. Then

$$\mu_p(\mathcal{A}) = \frac{\lambda([0, 1] \cdot \mathcal{A})}{\lambda([0, 1] \cdot \Delta_p^n)}, \quad \mathcal{A} \subset \Delta_p^n$$

is the normalized *cone measure* on Δ_p^n .

If ν_p denotes the p -normalized Lebesgue measure, i.e.

$$\nu_p(A) = \frac{\lambda(A)}{\lambda([0, 1] \cdot \Delta_p^n)}, \quad A \subset \mathbb{R}_+^n,$$

then the connection between ν_p and μ_p is given by

$$\nu_p(A) = n \int_0^\infty r^{n-1} \mu_p\left(\frac{\{x \in A : \|x\|_p = r\}}{r}\right) dr. \quad (12)$$

The proof of (12) follows directly for sets of the type $[a, b] \cdot \mathcal{A}$ with $0 < a < b < \infty$ and $\mathcal{A} \subset \Delta_p^n$ and is then finished by standard approximation arguments. The formula (12) may be generalized to the so-called *polar decomposition identity*, cf. [4],

$$\frac{\int_{\mathbb{R}_+^n} f(x) d\lambda(x)}{\lambda([0, 1] \cdot \Delta_p^n)} = n \int_0^\infty r^{n-1} \int_{\Delta_p^n} f(rx) d\mu_p(x) dr, \quad (13)$$

which holds for every $f \in L_1(\mathbb{R}_+^n)$.

The formula (13) allows to transfer immediately the results for the average surface best m -term approximation with respect to μ_p to the average volume approximation with respect to ν_p .

Proposition 3. *The identity*

$$\sigma_m^{p,q}(\nu_p) = \sigma_m^{p,q}(\mu_p) \cdot \frac{n}{n+1}$$

holds for all $0 < p \leq q \leq \infty$, all $n \geq 2$ and all $0 \leq m \leq n-1$.

Proof. We plug the function

$$f(x) = \sigma_m(x)_q \cdot \chi_{[0,1] \cdot \Delta_p^n}(x)$$

into (13) and obtain

$$\begin{aligned} \frac{\int_{[0,1] \cdot \Delta_p^n} \sigma_m(x)_q d\lambda(x)}{\lambda([0,1] \cdot \Delta_p^n)} &= \int_{[0,1] \cdot \Delta_p^n} \sigma_m(x)_q d\nu_p(x) \\ &= n \int_0^1 r^{n-1} \int_{\Delta_p^n} \sigma_m(rx)_q d\mu_p(x) dr = n \int_0^1 r^n dr \cdot \sigma_m^{p,q}(\mu_p), \end{aligned}$$

which gives the result. \square

Proposition 3 shows, that the ratio between approximation with respect to μ_p and ν_p is equal to $1 + 1/n$. This justifies our interest in measures on Δ_p^n . Furthermore, it shows that the quantities $\sigma_m^{p,q}(\nu_p)$ and $\sigma_m^{p,q}(\mu_p)$ behave asymptotically (i.e. for $n \rightarrow \infty$) very similarly.

Let $p = 2$ and let $\omega_1, \dots, \omega_n$ be independent normally distributed Gaussian random variables. Then

$$\varrho_2(\mathcal{A}) = \mu_2(\mathcal{A}) = \mathbb{P}\left(\frac{(|\omega_1|, \dots, |\omega_n|)}{(\sum_{j=1}^n \omega_j^2)^{1/2}} \in \mathcal{A}\right), \quad \mathcal{A} \subset \Delta_2^n.$$

As noted in [43], this relation may be generalized to all values of p with $0 < p < \infty$. Let $\omega_1, \dots, \omega_n$ be independent random variables on \mathbb{R}_+ each with density

$$c_p e^{-t^p}, \quad t \geq 0$$

with respect to the Lebesgue measure, where $c_p = \frac{p}{\Gamma(1/p)} = \frac{1}{\Gamma(1/p+1)}$.

Then, cf. [43, Lemma 1],

$$\mu_p(\mathcal{A}) = \mathbb{P}\left(\frac{(\omega_1, \dots, \omega_n)}{(\sum_{j=1}^n \omega_j^p)^{1/p}} \in \mathcal{A}\right), \quad \mathcal{A} \subset \Delta_p^n. \quad (14)$$

We shall fix $\omega_1, \dots, \omega_n$ to the end of this paper. Also the symbols \mathbb{E} and \mathbb{P} are always taken with respect to these variables.

2.1 The case $q = \infty$

In this section we deal with uniform approximation, i.e. with the case $q = \infty$. To be able to imitate the calculation (6), we shall need several tools, which are subject of Lemmas 4, 5 and 6. Our main result of this section (Theorem 7) then provides the estimate of $\sigma_m^{p,\infty}(\mu_p)$ from above for all m with $0 \leq m \leq n-1$. Furthermore, it is shown that in the range $0 \leq m \leq \varepsilon_p n$ this estimate is also optimal.

Lemma 4. *Let $0 < p < \infty$ and let $n \geq 2$ and $1 \leq m \leq n$ be natural numbers. Then*

$$\int_{\Delta_p^n} x_m^* d\mu_p(x) = \frac{\Gamma(n/p)}{\Gamma(n/p + 1/p)} \cdot \mathbb{E} x_m^*.$$

Furthermore, there are two positive real numbers C_p^1 and C_p^2 depending only on p , such that

$$C_p^1 \cdot \frac{\mathbb{E} x_m^*}{n^{1/p}} \leq \int_{\Delta_p^n} x_m^* d\mu_p(x) \leq C_p^2 \cdot \frac{\mathbb{E} x_m^*}{n^{1/p}}.$$

Proof. We put $f(x) = x_m^* e^{-x_1^p - \dots - x_n^p}$ and use the polar decomposition identity (13)

$$\begin{aligned} \frac{\int_{\mathbb{R}_+^n} x_m^* e^{-x_1^p - \dots - x_n^p} d\lambda(x)}{\lambda([0, 1] \cdot \Delta_p^n)} &= n \int_0^\infty r^{n-1} \int_{\Delta_p^n} (rx_m^*) \cdot e^{-(rx_1)^p - \dots - (rx_n)^p} d\mu_p(x) dr \\ &= n \int_0^\infty r^{n-1} \cdot r e^{-r^p} dr \int_{\Delta_p^n} x_m^* d\mu_p(x) \end{aligned}$$

or, equivalently,

$$\int_{\Delta_p^n} x_m^* d\mu_p(x) = \frac{\int_{\mathbb{R}_+^n} x_m^* e^{-x_1^p - \dots - x_n^p} d\lambda(x)}{\lambda([0, 1] \cdot \Delta_p^n) \cdot n \int_0^\infty r^n e^{-r^p} dr}. \quad (15)$$

The identity

$$\int_0^\infty r^n e^{-r^p} dr = \frac{\Gamma(n/p + 1/p)}{p},$$

follows by a simple substitution. Furthermore, we shall need the classical formula of Dirichlet for the volume of the unit ball $B_{\ell_p^n}$ of ℓ_p^n , cf. [21, p. 157],

$$\lambda([0, 1] \cdot \Delta_p^n) = \frac{\lambda(B_{\ell_p^n})}{2^n} = \frac{\Gamma(1/p + 1)^n}{\Gamma(n/p + 1)}.$$

This allows us to reformulate (15) as

$$\int_{\Delta_p^n} x_m^* d\mu_p(x) = \frac{\Gamma(n/p + 1) \mathbb{E} x_m^*}{c_p^n \cdot n/p \cdot \Gamma(n/p + 1/p) \Gamma(1/p + 1)^n} = \frac{\Gamma(n/p) \mathbb{E} x_m^*}{\Gamma(n/p + 1/p)}.$$

Finally, we use Stirling's formula (11) to estimate

$$\frac{n^{1/p} \cdot \Gamma(n/p)}{\Gamma(n/p + 1/p)} \leq C_p^1 \frac{n^{1/p} (n/p)^{n/p-1/2}}{(n/p + 1/p)^{n/p+1/p-1/2}} \leq C_p^2 \left(\frac{n}{n+1} \right)^{n/p+1/p-1/2} \leq C_p^3$$

and similarly for the estimate from below. \square

Lemma 5. *Let $\alpha \in \mathbb{R}$ and $\delta > 0$. Then*

$$\int_\delta^\infty u^\alpha e^{-u} du \leq \delta^\alpha e^{-\delta} \cdot \begin{cases} 1, & \text{if } \alpha \leq 0, \\ \frac{1}{1-\alpha/\delta}, & \text{if } \alpha > 0 \text{ and } \frac{\alpha}{\delta} < 1, \\ \left(\frac{\alpha}{\delta}\right)^\alpha \cdot \frac{\alpha/\delta}{1-\alpha/\delta}, & \text{if } \alpha > 0 \text{ and } \frac{\alpha}{\delta} > 1. \end{cases}$$

Proof. If $\alpha \leq 0$, we may estimate

$$\int_\delta^\infty u^\alpha e^{-u} du \leq \delta^\alpha \int_\delta^\infty e^{-u} du = \delta^\alpha e^{-\delta}.$$

If $0 < \alpha \leq 1$, we use partial integration and obtain

$$\int_\delta^\infty u^\alpha e^{-u} du = \delta^\alpha e^{-\delta} + \alpha \int_\delta^\infty u^{\alpha-1} e^{-u} du \leq \delta^\alpha e^{-\delta} (1 + \alpha \delta^{-1}).$$

This is smaller than

$$\delta^\alpha e^{-\delta} \left(1 + \frac{\alpha}{\delta} + \frac{\alpha^2}{\delta^2} + \dots\right) = \delta^\alpha e^{-\delta} \cdot \frac{1}{1 - \alpha/\delta}$$

if $\alpha/\delta < 1$ and smaller than

$$\delta^\alpha e^{-\delta} \frac{\alpha}{\delta} \left(1 + \frac{\delta}{\alpha} + \frac{\delta^2}{\alpha^2} + \dots\right) = \delta^\alpha e^{-\delta} \frac{\alpha}{\delta} \cdot \frac{1}{1 - \delta/\alpha}.$$

if $\alpha/\delta > 1$.

If $k - 1 < \alpha \leq k$ for some $k \in \mathbb{N}$, we iterate the partial integration and arrive at

$$\begin{aligned} \int_\delta^\infty u^\alpha e^{-u} du &\leq \delta^\alpha e^{-\delta} (1 + \alpha\delta^{-1} + \alpha(\alpha-1)\delta^{-2} + \dots + \alpha(\alpha-1)\dots(\alpha-k+1)\delta^{-k}) \\ &\leq \delta^\alpha e^{-\delta} \left(1 + \frac{\alpha}{\delta} + \frac{\alpha^2}{\delta^2} + \dots + \frac{\alpha^k}{\delta^k}\right) \\ &\leq \delta^\alpha e^{-\delta} \begin{cases} \frac{1}{1-\alpha/\delta}, & \text{if } \alpha/\delta < 1, \\ \left(\frac{\alpha}{\delta}\right)^{\alpha+1} \frac{1}{1-\delta/\alpha}, & \text{if } \alpha/\delta > 1. \end{cases} \end{aligned}$$

□

Lemma 6. *Let $0 < p < \infty$. Then there is a positive real number C_p , such that*

$$\mathbb{E} x_m^* \leq C_p \log^{1/p} \left(\frac{en}{m} \right)$$

for all $1 \leq m \leq n$.

Proof. We estimate

$$\begin{aligned} \mathbb{E} x_m^* &= \int_0^\infty \mathbb{P}(\omega_m^* > t) dt = \delta + \int_\delta^\infty \mathbb{P}(\omega_m^* > t) dt \\ &\leq \delta + \binom{n}{m} \int_\delta^\infty \mathbb{P}(\omega_1 > t, \omega_2 > t, \dots, \omega_m > t) dt \\ &= \delta + \binom{n}{m} \int_\delta^\infty \mathbb{P}(\omega_1 > t)^m dt. \end{aligned} \tag{16}$$

The parameter $\delta > \max(1, 3(1/p - 1))^{1/p}$ is to be chosen later on. We substitute $v = u^p$ and obtain

$$\mathbb{P}(\omega_1 > t) = c_p \int_t^\infty e^{-u^p} du = \frac{c_p}{p} \int_{t^p}^\infty v^{1/p-1} e^{-v} dv.$$

Using the first two estimates of Lemma 5 (recall that $t^p \geq \delta^p > \max(1, 3(1/p - 1))$), we arrive at

$$\mathbb{P}(\omega_1 > t) \leq C_p t^{1-p} e^{-t^p},$$

where C_p depends only on p . We plug this estimate into (16) and obtain

$$\mathbb{E} x_m^* \leq \delta + \binom{n}{m} (C_p)^m \int_\delta^\infty t^{m(1-p)} e^{-mt^p} dt. \tag{17}$$

If $p \geq 1$, then

$$\int_{\delta}^{\infty} t^{m(1-p)} e^{-mt^p} dt \leq \delta^{m(1-p)} \int_{\delta}^{\infty} e^{-mt^p} dt \leq \delta^{m(1-p)} \int_{m\delta^p}^{\infty} e^{-u} u^{1/p-1} du \leq e^{-m\delta^p}.$$

Altogether, we obtain

$$\mathbb{E} x_m^* \leq \delta + \binom{n}{m} (C_p)^m e^{-m\delta^p}.$$

Using $\binom{n}{m} \leq (\frac{en}{m})^m$ and choosing $\delta = C'_p \ln(\frac{en}{m})^{1/p}$ finishes the proof.

If $p < 1$, we use again the second estimate of Lemma 5

$$\begin{aligned} \int_{\delta}^{\infty} t^{m(1-p)} e^{-mt^p} dt &= \frac{1}{mp} \cdot m^{(1/p-1)(m+1)} \int_{m\delta^p}^{\infty} u^{(1/p-1)(m+1)} e^{-u} du \\ &\leq \frac{1}{mp} \cdot \delta^{(1-p)(m+1)} e^{-m\delta^p} \cdot \frac{1}{1 - \frac{2(1/p-1)}{\delta^p}} \leq C'_p \delta^{(1-p)(m+1)} e^{-m\delta^p}. \end{aligned}$$

Using (17) and $\binom{n}{m} \leq (\frac{en}{m})^m$ again, we get

$$\begin{aligned} \mathbb{E} x_1^* &\leq \delta + \exp(-m\delta^p + m \ln(en/m) + (1-p)(m+1) \ln \delta + m \ln C_p + \ln C'_p) \\ &\leq \delta + \exp[-m(\delta^p + C_p \ln(en/m) + 2(1-p) \ln \delta)] \end{aligned}$$

The choice $\delta = C'_p \ln(\frac{en}{m})^{1/p}$ with C'_p large enough ensures, that

$$\frac{\delta^p}{2} \geq C_p \ln(en/m) \quad \text{and} \quad \frac{\delta^p}{2} \geq 2(1-p) \ln \delta$$

and finishes the proof. \square

The following theorem gives the basic estimates of $\sigma_m^{p,\infty}(\mu_p)$.

Theorem 7. *Let $0 < p \leq \infty$ and let $n \geq 2$.*

(i) *Let $0 \leq m \leq n-1$. Then*

$$\sigma_m^{p,\infty}(\mu_p) \leq C_p \left[\frac{\log\left(\frac{en}{m+1}\right)}{n} \right]^{1/p}. \quad (18)$$

(ii) *There is a number $0 < \varepsilon_p < 1$, such that for $0 \leq m \leq \varepsilon_p n$ the following estimate holds*

$$\sigma_m^{p,\infty}(\mu_p) \geq C_p \left[\frac{\log\left(\frac{en}{m+1}\right)}{n} \right]^{1/p}. \quad (19)$$

Proof. Lemma 4 and Lemma 6 imply immediately the first part of the theorem if $p < \infty$. If $p = \infty$, the proof is trivial.

The proof of the second part is divided into two steps.

Step 1. We start first with the case $m = 0$.

If $p = \infty$, then $x_1^* = 1$ for all $x \in \Delta_p^n$ and the proof is trivial. Let us therefore assume, that $p < \infty$. According to Lemma 4, we have to estimate $\mathbb{E} x_1^*$ from below.

This was done in [43, Lemma 2]. We include a slightly different proof for readers convenience. For every $t_0 > 0$, it holds

$$\mathbb{E} x_1^* \geq t_0 \mathbb{P}(x_1^* > t_0) = t_0 \mathbb{P}\left(\max_{1 \leq j \leq n} x_j > t_0\right) \geq t_0 [n \mathbb{P}(x_1 > t_0) - \binom{n}{2} \mathbb{P}(x_1 > t_0)^2].$$

We define t_0 by $\mathbb{P}(x_1 > t_0) = \frac{1}{n}$ and obtain $\mathbb{E} x_1^* \geq t_0/2$.

From the simple estimate

$$\frac{c_p}{p} \int_{T^p}^{\infty} u^{1/p-1} e^{-u} du \geq C_p e^{-2T^p}, \quad T > 1,$$

it follows, that there is a positive real number $\gamma_p > 0$, such that

$$\mathbb{P}(x_1 > \gamma_p (\log(en))^{1/p}) \geq 1/n.$$

This gives $t_0 \geq \gamma_p (\log(en))^{1/p}$ and $\mathbb{E} x_1^* \geq C_p (\log(en))^{1/p}$.

Step 2. Let $0 \leq m \leq \varepsilon_p n$, where $\varepsilon_p > 0$ will be chosen later on.

We shall use the inequality

$$\frac{1}{m} \sum_{j=1}^m \log^{1/p} \left(\frac{en}{j} \right) \leq C_p \log^{1/p} \left(\frac{en}{m} \right), \quad 1 \leq m \leq n, \quad (20)$$

which follows by direct calculation for $p = 1$, by Hölder's inequality for $1 < p < \infty$ and by replacing the sum by the corresponding integral and integration by parts if $0 < p < 1$.

We denote

$$\|x\|_{(m)} = \frac{1}{m} \sum_{j=1}^m x_j^*.$$

By Lemma 6 and (20),

$$\mathbb{E} \|x\|_{(m)} = \frac{1}{m} \sum_{j=1}^m \mathbb{E} x_j^* \leq \frac{C_p}{m} \sum_{j=1}^m \log^{1/p} \left(\frac{en}{j} \right) \leq C_p^1 \log^{1/p} \left(\frac{en}{m} \right). \quad (21)$$

To estimate $\mathbb{E} \|x\|_{(m)}$ from below, we assume that $1 \leq m \leq n$ and that n/m is an integer (otherwise one has to slightly modify the argument at the cost of the constants involved). We partition the set $\{1, \dots, n\} = A_1 \cup \dots \cup A_m$, where each one of the disjoint sets A_j has n/m elements. Then we have

$$\|x\|_{(m)} \geq \frac{1}{m} \sum_{j=1}^m \max_{l \in A_j} x_l$$

and by the first step we obtain

$$\mathbb{E} \|x\|_{(m)} \geq \frac{1}{m} \sum_{j=1}^m \mathbb{E} \max_{l \in A_j} x_l \geq C_p^2 \log^{1/p} \left(\frac{en}{m} \right). \quad (22)$$

Let $N_p < 1/\varepsilon_p$ be a natural number to be chosen later on. Combining (21) with (22) gives finally

$$\begin{aligned}\mathbb{E} x_m^* &\geq \frac{1}{N_p m} \sum_{k=m}^{N_p m} \mathbb{E} x_k^* \geq \mathbb{E} \|x\|_{(N_p m)} - \frac{1}{N_p} \mathbb{E} \|x\|_{(m)} \\ &\geq C_p^2 \log^{1/p} \left(\frac{en}{N_p m} \right) - \frac{C_p^1}{N_p} \log^{1/p} \left(\frac{en}{m} \right) \\ &= \log^{1/p} \left(\frac{en}{m} \right) \left\{ C_p^2 \left[1 - \frac{\log(N_p)}{\log \left(\frac{en}{m} \right)} \right]^{1/p} - \frac{C_p^1}{N_p} \right\}.\end{aligned}$$

An appropriate choice of N_p and ε_p (i.e. $N_p > 2^{1/p} C_p^1 / C_p^2$ and $\varepsilon_p < \min(1/N_p, e/N_p^2)$) with

$$C_p^2 \left[1 - \frac{\log(N_p)}{\log \left(\frac{e}{\varepsilon_p} \right)} \right]^{1/p} - \frac{C_p^1}{N_p} > 0$$

gives the result. \square

Remark 1. (i) Theorem 7 provides basic estimates of average best m -term widths $\sigma_m^{p,\infty}(\mu_p)$. In the case $m = 0$ a stronger result on concentration of μ_p was obtained already in [43, Theorem 3 and Remark 2]. It would be certainly of interest to obtain a similar statement also for other values of $m > 0$, but this would go beyond the scope of this paper and we leave this direction open for further study.

- (ii) Theorem 7 may be interpreted in the sense of the discussion after formula (7). Namely, the average coordinate of $x \in \Delta_p^n$ is $n^{-1/p}$. Theorem 7 shows, that the average value of the largest coordinate is only slightly larger (namely $c[\ln(en)]^{1/p}$ times larger). In this sense, the average point of Δ_p^n is only slightly modified (and properly normalized) white noise.
- (iii) Using the interpolation formula (4), one may immediately extend this result to all $0 < p \leq q < \infty$. But we shall see later on, that in the case $q < \infty$, one may prove slightly better estimates.
- (iv) The behavior of $\sigma_m^{p,\infty}(\mu_p)$ was studied in detail in [28, Example 10] for $p = 2$. It was shown that if x_i are independent $N(0, 1)$ Gaussian random variables and $m \leq n/2 + 1$, then

$$c \sqrt{\ln \frac{2n}{m}} \leq \mathbb{E} x_m^* \leq C \sqrt{\ln \frac{2n}{m}},$$

where c and C are absolute positive constants. Furthermore, if $m \geq n/2 + 1$, then

$$\sqrt{\frac{\pi}{2}} \frac{n - m + 1}{n + 1} \leq \mathbb{E} x_m^* \leq \sqrt{2\pi} \frac{n - m + 1}{n}.$$

- (v) The method used in the proof of the second part of Theorem 7 may be found for example in [27].

2.2 The case $q < \infty$

We discuss briefly also the case when $q < \infty$. It turns out, that in this case the logarithmic term disappears. We do not go much into details and restrict ourselves to the case $m = 0$.

Proposition 8. *Let $n \geq 2$ and $0 < p \leq q < \infty$. Then*

$$(i) \quad C_{p,q}^1 n^{1/q} \leq \mathbb{E} \|x\|_q \leq C_{p,q}^2 n^{1/q},$$

(ii)

$$C_{p,q}^1 \cdot \frac{\mathbb{E} \|x\|_q}{n^{1/p}} \leq \sigma_0^{p,q}(\mu_p) = \int_{\Delta_p^n} \|x\|_q d\mu_p(x) \leq C_{p,q}^2 \cdot \frac{\mathbb{E} \|x\|_q}{n^{1/p}}$$

and

$$(iii) \quad C_{p,q}^1 n^{1/q-1/p} \leq \sigma_0^{p,q}(\mu_p) \leq C_{p,q}^2 n^{1/q-1/p},$$

where in all these estimates C_p^1 and C_p^2 are positive real numbers depending only on p .

Proof. (i) The following two inequalities may be easily proved by Hölder's and Minkowski inequality.

$$\begin{aligned} \left(\sum_{j=1}^n (\mathbb{E} x_j^q) \right)^{1/q} &\leq \mathbb{E} \left(\sum_{j=1}^n x_j^q \right)^{1/q} \leq \left(\sum_{j=1}^n \mathbb{E} x_j^q \right)^{1/q}, \quad q \geq 1, \\ \left(\sum_{j=1}^n \mathbb{E} x_j^q \right)^{1/q} &\leq \mathbb{E} \left(\sum_{j=1}^n x_j^q \right)^{1/q} \leq \left(\sum_{j=1}^n (\mathbb{E} x_j^q) \right)^{1/q}, \quad q \leq 1. \end{aligned}$$

This gives for $q \geq 1$

$$\mathbb{E} \|x\|_q \leq n^{1/q} (\mathbb{E} x_j^q)^{1/q} \quad \text{and} \quad \mathbb{E} \|x\|_q \geq n^{1/q} \mathbb{E} x_j$$

and for $q \leq 1$

$$\mathbb{E} \|x\|_q \leq n^{1/q} \mathbb{E} x_j \quad \text{and} \quad \mathbb{E} \|x\|_q \geq n^{1/q} (\mathbb{E} x_j^q)^{1/q}.$$

Let us note, that the value of $\mathbb{E} x_j$ and $(\mathbb{E} x_j^q)^{1/q}$ does not depend on n , only on p and q .

(ii) The proof of the second part resembles very much the proof of Lemma 4 and is left to the reader.

(iii) The last point follows immediately from (i) and (ii). \square

Remark 2. A similar statement to Proposition 8 is included in [43, Lemma 2, point 4].

3 Normalized surface measure

In this section we study the average best m -term widths for another classical measure on Δ_p^n , namely the normalized Hausdorff measure, cf. Definition 9. Intuitively, this measure gives more weight to those areas, where one or more components of $x \in \Delta_p^n$ are close to zero. It turns out, that this is really the case - with the mathematical formulation given in Lemma 10 below. This relation is then used together with Lemma 11 in Theorem 12 to provide estimates of $\sigma_0^{p,\infty}(\varrho_p)$ from above.

Definition 9. Let $n \geq 2$ be a natural number. We denote by

$$\varrho_p(\mathcal{A}) = \frac{\mathcal{H}(\mathcal{A})}{\mathcal{H}(\Delta_p^n)}, \quad \mathcal{A} \subset \Delta_p^n$$

the normalized $n - 1$ dimensional Hausdorff measure on Δ_p^n .

Let us mention, that for $p \in \{1, 2, \infty\}$ the measure ϱ_p coincides with μ_p . The following lemma provides a relationship between the normalized surface measure ϱ_p and the cone measure μ_p . For $p \geq 1$, it was given by [38]. We follow closely their approach and it turns out, that it may be generalized also to the non-convex case of $0 < p < 1$.

Lemma 10. *Let $0 < p < \infty$ and $n \geq 2$. Then ϱ_p is an absolutely continuous measure with respect to μ_p and for μ_p almost every $x \in \Delta_p^n$ it holds*

$$\frac{d\varrho_p}{d\mu_p}(x) = \frac{n\lambda([0, 1] \cdot \Delta_p^n)}{\mathcal{H}(\Delta_p^n)} \left\| \nabla(\|\cdot\|_p)(x) \right\|_2 = c_{p,n}^{-1} \left(\sum_{i=1}^n x_i^{2p-2} \right)^{1/2},$$

where

$$c_{p,n} = \int_{\Delta_p^n} \left(\sum_{i=1}^n x_i^{2p-2} \right)^{1/2} d\mu_p(x)$$

is the normalizing constant.

Proof. The proof imitates the proof of [38, Lemma 1 and Lemma 2], where the statement was proven for $1 \leq p < \infty$. Hence, we may assume, that $0 < p < 1$. First, we introduce some notation.

We fix $x = (x_1, \dots, x_n) \in \Delta_p^n$, such that

- the mapping $y \rightarrow \|y\|_p$ is differentiable at x ,
- x is a density point of \mathcal{H} , i.e.

$$\lim_{\varepsilon \rightarrow 0+} \frac{\mathcal{H}(B(x, \varepsilon) \cap \Delta_p^n)}{\varepsilon^{n-1} V_{n-1}} = 1, \quad (23)$$

where V_{n-1} denotes the Lebesgue volume of the $n - 1$ dimensional Euclidean unit ball.

- $x_i > 0$ for all $i = 1, \dots, n$.

Obviously, ϱ_p -almost every $x \in \Delta_p^n$ satisfies all the three properties (we refer for example to [34, Theorem 16.2] for the second one).

Furthermore, we put $z := \nabla(\|\cdot\|_p)(x)$. This means, that

$$\|x + y\|_p = 1 + \langle z, y \rangle + r(y), \quad (24)$$

where

$$\theta(\delta) := \sup \left\{ \frac{|r(y)|}{\|y\|_2} : 0 < \|y\|_2 \leq \delta \right\}, \quad \delta > 0$$

tends to zero if δ tends to zero. Using (24) for $y = \delta x$, one observes, that $\langle z, x \rangle = 1$. We denote by $H = x + z^\perp$ the tangent hyperplane to Δ_p^n at x . Let us note, that for $0 < p < 1$ the set $\mathbb{R}_+^n \setminus [0, 1] \cdot \Delta_p^n = [1, \infty) \cdot \Delta_p^n$ is convex. Next, we show, that $\langle z, y \rangle \geq 1$ for every $y \in [1, \infty) \cdot \Delta_p^n$. Indeed,

$$\begin{aligned} 1 &\leq \|x + \lambda(y - x)\|_p = 1 + \langle z, \lambda(y - x) \rangle + r(\lambda(y - x)) \\ &= 1 - \lambda + \lambda \langle z, y \rangle + r(\lambda(y - x)) \end{aligned}$$

Dividing by $\lambda > 0$ and letting $\lambda \rightarrow 0$ gives the statement.

The proof of the lemma is based on the following two inclusions, namely

$$[0, 1] \cdot \left(B(x, \varepsilon(1 - \theta(\varepsilon))) \cap H \right) \subset [0, 1] \cdot \left(B(x, \varepsilon) \cap \Delta_p^n \right) \quad (25)$$

and

$$[0, 1] \cdot \left(B(x, \varepsilon) \cap \Delta_p^n \right) \subset [0, 1 + \varepsilon\theta(\varepsilon)] \cdot \left(B(x, \varepsilon(1 + \theta(\varepsilon)\|x\|_2)) \cap H \right), \quad (26)$$

which hold for all $\varepsilon > 0$ small enough.

First, we prove (25). To given $0 \leq s \leq 1$ and $v \in B(x, \varepsilon(1 - \theta(\varepsilon))) \cap H$ we need to find $0 \leq t \leq 1$ and $w \in B(x, \varepsilon) \cap \Delta_p^n$, such that $sv = tw$. To do this, we set

$$w := \frac{v}{\|v\|_p} \in \Delta_p^n \quad \text{and} \quad t := s\|v\|_p.$$

We need to show, that $t \leq 1$ and $\|x - w\|_2 \leq \varepsilon$.

We choose $0 < \varepsilon \leq \min_i x_i$. Then

$$x_i \leq |x_i - v_i| + v_i \leq \|x - v\|_2 + v_i \leq \varepsilon + v_i$$

for every $i = 1, \dots, n$, which implies, that $v_i \geq 0$ and $v \in \mathbb{R}_+^n$. From $v \in H$ and $v \in \mathbb{R}_+^n$ we deduce, that $\|v\|_p \leq 1$. Hence $t = s\|v\|_p \leq \|v\|_p \leq 1$.

Next, we write

$$\begin{aligned} \|x - w\|_2 &= \left\| x - \frac{v}{\|v\|_p} \right\|_2 \leq \|x - v\|_2 + \left\| v - \frac{v}{\|v\|_p} \right\|_2 \\ &\leq \varepsilon(1 - \theta(\varepsilon)) + \|v\|_2 \cdot \frac{1 - \|v\|_p}{\|v\|_p} \leq \varepsilon(1 - \theta(\varepsilon)) + 1 - \|v\|_p \\ &= \varepsilon(1 - \theta(\varepsilon)) + 1 - \{1 + \langle v - x, z \rangle + r(v - x)\} \\ &= \varepsilon(1 - \theta(\varepsilon)) + r(v - x) \leq \varepsilon. \end{aligned}$$

Next, we prove (26). We need to find to given $0 \leq t \leq 1$ and $w \in B(x, \varepsilon) \cap \Delta_p^n$ some $0 \leq s \leq 1 + \varepsilon\theta(\varepsilon)$ and $v \in B(x, \varepsilon(1 + \theta(\varepsilon)\|x\|_2)) \cap H$, such that $tw = sv$. We put

$$s := t\langle w, z \rangle \quad \text{and} \quad v := \frac{w}{\langle w, z \rangle}.$$

Let us recall, that we have shown above, that $w \in \Delta_p^n$ implies that $\langle w, z \rangle \geq 1$.

Of course, $tw = sv$ and $v \in H$ (as $\langle v, z \rangle = 1$). Hence, it remains to show, that $s \leq 1 + \varepsilon\theta(\varepsilon)$ and $\|v - x\|_2 \leq \varepsilon(1 + \theta(\varepsilon)\|x\|_2)$.

The application of (24) gives

$$1 = \|w\|_p = \|x + (w - x)\|_p = 1 + \langle w - x, z \rangle + r(w - x),$$

which again forces $\langle w, z \rangle \leq 1 + \varepsilon\theta(\varepsilon)$. Then $s = t\langle w, z \rangle \leq \langle w, z \rangle \leq 1 + \varepsilon\theta(\varepsilon)$.

Finally, we write

$$\begin{aligned} \|v - x\|_2 &= \left\| \frac{w}{\langle w, z \rangle} - x \right\|_2 \leq \left\| \frac{w}{\langle w, z \rangle} - \frac{x}{\langle w, z \rangle} \right\|_2 + \left\| \frac{x}{\langle w, z \rangle} - x \right\|_2 \\ &\leq \frac{\|w - x\|_2}{\langle w, z \rangle} + \|x\|_2 \frac{\langle w, z \rangle - 1}{\langle w, z \rangle} \leq \varepsilon + \varepsilon\theta(\varepsilon)\|x\|_2. \end{aligned}$$

Equipped with (25) and (26), we may finish the proof of the lemma. We write

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \frac{\varrho_p(B(x, \varepsilon) \cap \Delta_p^n)}{\mu_p(B(x, \varepsilon) \cap \Delta_p^n)} &= \lim_{\varepsilon \rightarrow 0} \frac{\mathcal{H}(B(x, \varepsilon) \cap \Delta_p^n)}{\mathcal{H}(\Delta_p^n)} \cdot \frac{\varepsilon^{n-1}V_{n-1}}{\varepsilon^{n-1}V_{n-1}} \cdot \frac{\lambda([0, 1] \cdot \Delta_p^n)}{\lambda([0, 1] \cdot [B(x, \varepsilon) \cap \Delta_p^n])} \\ &= \frac{\lambda([0, 1] \cdot \Delta_p^n)}{\mathcal{H}(\Delta_p^n)} \cdot \lim_{\varepsilon \rightarrow 0} \frac{\varepsilon^{n-1}V_{n-1}}{\lambda([0, 1] \cdot [B(x, \varepsilon) \cap \Delta_p^n])}, \end{aligned} \quad (27)$$

where we have used (23). As the perpendicular distance between zero and H is equal to $1/\|z\|_2$, we observe, that

$$\text{vol}(B(x, a) \cap H) = \frac{a^{n-1}V_{n-1}}{n\|z\|_2}$$

holds for every $a > 0$. Using this, we get from (25) and (26)

$$\begin{aligned} \lambda\left([0, 1] \cdot \left(B(x, \varepsilon(1 - \theta(\varepsilon))) \cap H\right)\right) &= \frac{[\varepsilon(1 - \theta(\varepsilon))]^{n-1}V_{n-1}}{n\|z\|_2} \\ &\leq \lambda\left([0, 1] \cdot \left(B(x, \varepsilon) \cap \Delta_p^n\right)\right) \\ &\leq \lambda\left([0, 1 + \varepsilon\theta(\varepsilon)] \cdot \left(B(x, \varepsilon(1 + \theta(\varepsilon)\|x\|_2)) \cap H\right)\right) \\ &= [1 + \varepsilon\theta(\varepsilon)]^n \cdot \frac{[\varepsilon(1 + \theta(\varepsilon)\|x\|_2)]^{n-1}V_{n-1}}{n\|z\|_2}. \end{aligned}$$

Combining these estimates with (27) gives the result. \square

Following lemma is analogous to Lemma 4 and reduces the calculation of $\sigma_0^{p, \infty}(\varrho_p)$ to inequalities for the estimated values of functions of the random variables x_1, \dots, x_n .

Lemma 11. Let $0 < p < \infty$. There exists two positive real numbers C_p^1 and C_p^2 , such that

$$\begin{aligned} C_p^1 \cdot \frac{\mathbb{E} x_1^* \left(\sum_{i=1}^n x_i^{2p-2} \right)^{1/2}}{\mathbb{E} \left(\sum_{i=1}^n x_i^{2p-2} \right)^{1/2}} \cdot n^{-1/p} &\leq \sigma_0^{p,\infty}(\varrho_p) = \int_{\Delta_p^n} x_1^* d\varrho_p \\ &= \frac{\int_{\Delta_p^n} x_1^* \left(\sum_{i=1}^n x_i^{2p-2} \right)^{1/2} d\mu_p(x)}{\int_{\Delta_p^n} \left(\sum_{i=1}^n x_i^{2p-2} \right)^{1/2} d\mu_p(x)} \leq C_p^2 \frac{\mathbb{E} x_1^* \left(\sum_{i=1}^n x_i^{2p-2} \right)^{1/2}}{\mathbb{E} \left(\sum_{i=1}^n x_i^{2p-2} \right)^{1/2}} \cdot n^{-1/p} \end{aligned} \quad (28)$$

for all $n \geq 2$.

Proof. Only the inequalities need a proof. It resembles the proof of Lemma 4 and is again based on the polar decomposition formula (13).

We plug the functions

$$f_1(x) = x_1^* \left(\sum_{i=1}^n x_i^{2p-2} \right)^{1/2} e^{-x_1^p - \dots - x_n^p} \quad \text{and} \quad f_2(x) = \left(\sum_{i=1}^n x_i^{2p-2} \right)^{1/2} e^{-x_1^p - \dots - x_n^p}$$

into (13) and obtain

$$\begin{aligned} \sigma_0^{p,\infty}(\varrho_p) &= \frac{\int_{\mathbb{R}_+^n} f_1(x) dx \cdot \int_0^\infty r^{n+p-2} e^{-r^p} dr}{\int_{\mathbb{R}_+^n} f_2(x) dx \cdot \int_0^\infty r^{n+p-1} e^{-r^p} dr} \\ &= \frac{\mathbb{E} x_1^* \left(\sum_{i=1}^n x_i^{2p-2} \right)^{1/2}}{\mathbb{E} \left(\sum_{i=1}^n x_i^{2p-2} \right)^{1/2}} \cdot \frac{\Gamma(n/p + 1 - 1/p)}{\Gamma(n/p + 1)}. \end{aligned}$$

By Stirling's formula, the last expression is equivalent to $n^{-1/p}$ with constants of equivalence depending only on p . \square

Theorem 12. Let $0 < p < \infty$. Then there is a positive real number C_p , such that

$$\sigma_0^{p,\infty}(\varrho_p) \leq C_p \left[\frac{\log(n+1)}{n} \right]^{1/p} \quad (29)$$

for all $n \geq 2$.

Proof. We define a probability measure $\alpha_{p,n}$ on \mathbb{R}_+^n by the density

$$\tilde{c}_{p,n}^{-1} \cdot \left(\sum_{i=1}^n x_i^{2p-2} \right)^{1/2} e^{-x_1^p - \dots - x_n^p}, \quad \tilde{c}_{p,n} := \int_{\mathbb{R}_+^n} \left(\sum_{i=1}^n x_i^{2p-2} \right)^{1/2} e^{-x_1^p - \dots - x_n^p} dx$$

with respect to the Lebesgue measure. Let us note, that due to the inequality

$$\left(\sum_{i=1}^n x_i^{2p-2} \right)^{1/2} \leq \sum_{i=1}^n x_i^{p-1}$$

the integral in the definition of $\tilde{c}_{p,n}$ really converges and $\alpha_{p,n}$ is well defined.

According to Lemma 11, we need to estimate

$$\int_{\mathbb{R}_+^n} x_1^* d\alpha_{p,n}(x).$$

We calculate for $\delta > 1$, which is to be chosen later on,

$$\begin{aligned} \int_{\mathbb{R}_+^n} x_1^* d\alpha_{p,n}(x) &= \int_0^\infty \alpha_{p,n}(x_1^* > t) dt \leq \delta + \int_\delta^\infty \alpha_{p,n}(x_1^* > t) dt \\ &\leq \delta + n \int_\delta^\infty \alpha_{p,n}(x_1 > t) dt. \end{aligned}$$

We write $x' = (x_2, \dots, x_n) \in \mathbb{R}_+^{n-1}$. Then

$$\begin{aligned} \alpha_{p,n}(x_1 > t) &= \tilde{c}_{p,n}^{-1} \int_t^\infty e^{-x_1^p} \int_{\mathbb{R}_+^{n-1}} \left(\sum_{i=1}^n x_i^{2p-2} \right)^{1/2} e^{-x_2^p - \dots - x_n^p} dx' dx_1 \\ &\leq \tilde{c}_{p,n}^{-1} \int_t^\infty e^{-x_1^p} \int_{\mathbb{R}_+^{n-1}} \left[x_1^{p-1} + \left(\sum_{i=2}^n x_i^{2p-2} \right)^{1/2} \right] e^{-x_2^p - \dots - x_n^p} dx' dx_1 \\ &= \tilde{c}_{p,n}^{-1} \int_t^\infty e^{-x_1^p} x_1^{p-1} dx_1 \cdot \int_{\mathbb{R}_+^{n-1}} e^{-x_2^p - \dots - x_n^p} dx' \\ &\quad + \tilde{c}_{p,n}^{-1} \int_t^\infty e^{-x_1^p} dx_1 \cdot \int_{\mathbb{R}_+^{n-1}} \left(\sum_{i=2}^n x_i^{2p-2} \right)^{1/2} e^{-x_2^p - \dots - x_n^p} dx' \\ &:= I_1 + I_2. \end{aligned}$$

The inequality

$$\begin{aligned} c_p^n \tilde{c}_{p,n} &= c_p^n \int_{\mathbb{R}_+^n} \left(\sum_{i=1}^n x_i^{2p-2} \right)^{1/2} e^{-x_1^p - \dots - x_n^p} dx \\ &\geq c_p^n \int_{\mathbb{R}_+^n} \left(\sum_{i=2}^n x_i^{2p-2} \right)^{1/2} e^{-x_1^p - \dots - x_n^p} dx \\ &= c_p^n \int_0^\infty e^{-x_1^p} dx_1 \int_{\mathbb{R}_+^{n-1}} \left(\sum_{i=2}^n x_i^{2p-2} \right)^{1/2} e^{-x_2^p - \dots - x_n^p} dx' = c_p^{n-1} \tilde{c}_{p,n-1} \end{aligned} \tag{30}$$

shows, that

$$I_1 = \frac{c_p \int_t^\infty x_1^{p-1} e^{-x_1^p} dx_1}{c_p^n \tilde{c}_{p,n}} \leq \frac{c_p \int_t^\infty x_1^{p-1} e^{-x_1^p} dx_1}{c_p \tilde{c}_{p,1}} = \tilde{c}_{p,1}^{-1} \cdot \frac{e^{-t^p}}{p}.$$

Using (30) again, we get also

$$I_2 = \tilde{c}_{p,n}^{-1} \cdot \tilde{c}_{p,n-1} \int_t^\infty e^{-x_1^p} dx_1 \leq c_p \int_t^\infty e^{-x_1^p} dx_1 = \frac{c_p}{p} \cdot \int_{t^p}^\infty s^{1/p-1} e^{-s} ds.$$

If $p \geq 1$, we get

$$I_1 + I_2 \leq C_p e^{-t^p}, \quad t > 1 \quad (31)$$

and

$$\int_{\mathbb{R}_+^n} x_1^* d\alpha_{p,n}(x) \leq \delta + C_p n \int_\delta^\infty e^{-t^p} dt \leq \delta + C'_p n e^{-\delta^p}.$$

By choosing $\delta = C_p \log(n+1)^{1/p}$, we get the result.

If $p < 1$, we use the second estimate of Lemma 5 and replace (31) with

$$I_1 + I_2 \leq C_p t^{1-p} e^{-t^p}, \quad t > t_0$$

for $t_0 > 1$ large enough and the result again follows by the choice of δ . □

Remark 3. (i) Theorem 12 shows, that the average size of the largest coordinate of $x \in \Delta_p^n$ taken with respect to the normalized Hausdorff measure is again only slightly larger than $n^{-1/p}$. Hence, also in this case, the typical element of Δ_p^n seems to be far from being sparse and resembles rather properly normalized white noise in the sense described in Introduction.

(ii) Using interpolation inequality (4), one may again obtain a similar estimate also for $0 < p \leq q < \infty$, namely

$$\sigma_0^{p,q}(\varrho_p) \leq C_{p,q} \left[\frac{\log(n+1)}{n} \right]^{1/p-1/q}.$$

It would be probably possible to avoid the logarithmic terms and provide improved estimates also for $m > 0$, but we shall not go into this direction. Our main aim of this section was to show, that normalized Hausdorff measure does not prefer sparse (or nearly sparse) vectors, and this was clearly demonstrated by Theorem 12.

4 Tensor product measures

As discussed already in the Introduction and proved in Theorem 7 and Theorem 12, the average vectors of Δ_p^n with respect to the cone measure μ_p and with respect to surface measure ϱ_p behave “badly” meaning that (roughly speaking) many of their coordinates are approximately of the same size. As promised before, we shall now introduce a new class of measures, for which the random vector behaves in a completely different way. These measures are defined through their density with respect to the cone measure μ_p . This density has a strong singularity near the points with vanishing coordinates.

Definition 13. Let $0 < p < \infty$, $\beta > -1$ and $n \geq 2$. Then we define the probability measure $\theta_{p,\beta}$ on Δ_p^n by

$$\frac{d\theta_{p,\beta}}{d\mu_p}(x) = c_{p,\beta}^{-1} \cdot \prod_{i=1}^n x_i^\beta, \quad x \in \Delta_p^n, \quad (32)$$

where

$$c_{p,\beta} = \int_{\Delta_p^n} \prod_{i=1}^n x_i^\beta d\mu_p(x). \quad (33)$$

Remark 4. (i) If $0 > \beta > -1$, then (32) defines the density of $\theta_{p,\beta}$ with respect to μ_p only for points, where $x_i \neq 0$ for all $i = 1, \dots, n$. That means, that this density is defined μ_p -almost everywhere. The definition is then complemented by the statement, that $\theta_{p,\beta}$ is absolutely continuous with respect to μ_p .

(ii) We shall see later on, that the condition $\beta > -1$ ensures, that (33) is finite.

(iii) It was observed already in [4], that the measures $\theta_{p,\beta}$ allow a formula similar to (14). We plug the function $f(x) = \chi_{[0,\infty) \cdot \mathcal{A}} \prod_{i=1}^n x_i^\beta e^{-\|x\|_p^p}$ into (13), where \mathcal{A} is any μ_p -measurable subset of Δ_p^n , and obtain

$$\int_{[0,\infty) \cdot \mathcal{A}} \prod_{i=1}^n x_i^\beta e^{-\|x\|_p^p} d\lambda(x) = \lambda([0,1] \cdot \Delta_p^n) \cdot n \cdot \int_0^\infty r^{n-1+n\beta} e^{-r^p} dr \cdot \int_{\mathcal{A}} \prod_{i=1}^n x_i^\beta d\mu_p(x).$$

We use a similar formula also for $\mathcal{A} = \Delta_p^n$, which leads to

$$\int_{\mathcal{A}} 1 d\theta_{p,\beta} = \frac{\int_{\mathcal{A}} \prod_{i=1}^n x_i^\beta d\mu_p(x)}{\int_{\Delta_p^n} \prod_{i=1}^n x_i^\beta d\mu_p(x)} = \frac{\int_{[0,\infty) \cdot \mathcal{A}} \prod_{i=1}^n x_i^\beta e^{-\|x\|_p^p} dx}{\int_{\mathbb{R}_+^n} \prod_{i=1}^n x_i^\beta e^{-\|x\|_p^p} dx}.$$

Let $\omega' = (\omega'_1, \dots, \omega'_n)$ be a vector with independent identically distributed components with respect to the density $c_{p,\beta} t^\beta e^{-t^p}$, $t > 0$, where $c_{p,\beta}^{-1} = \int_0^\infty t^\beta e^{-t^p} dt$ is a normalizing constant. Up to a simple substitution, this is the well known *gamma distribution*. We observe that the distribution of random points with respect to $\theta_{p,\beta}$ equals to the distribution of ℓ_p^n normalized vectors ω' , i.e.

$$\theta_{p,\beta}(\mathcal{A}) = \mathbb{P}\left(\frac{(\omega'_1, \dots, \omega'_n)}{(\sum_{j=1}^n (\omega'_j)^p)^{1/p}} \in \mathcal{A}\right), \quad \mathcal{A} \subset \Delta_p^n. \quad (34)$$

(iv) Of course, the same procedure might be considered also for other distributions. We leave this to future work. We also refer to the discussion on the recent work of Gribonval, Cevher, and Davies [29] in the Introduction.

Lemma 14. Let $0 < p < \infty$, $\beta > -1$ and $n \geq 2$.

(i) Let $1 \leq m \leq n$. Then

$$\sigma_{m-1}^{p,\infty}(\theta_{p,\beta}) = \int_{\Delta_p^n} x_m^* d\theta_{p,\beta} = \frac{\mathbb{E} x_m^* \prod_{i=1}^n x_i^\beta}{\mathbb{E} \prod_{i=1}^n x_i^\beta} \cdot \frac{\Gamma(n(\beta+1)/p)}{\Gamma(n(\beta+1)/p + 1/p)}.$$

(ii)

$$\mathbb{E} \prod_{i=1}^n x_i^\beta = \left[\frac{c_p}{p} \cdot \Gamma((\beta+1)/p) \right]^n.$$

Proof. The proof of the first part follows again by (13), this time used for the functions

$$f_1(x) = x_m^* \left(\prod_{i=1}^n x_i^\beta \right) e^{-x_1^p - \dots - x_n^p} \quad \text{and} \quad f_2(x) = \left(\prod_{i=1}^n x_i^\beta \right) e^{-x_1^p - \dots - x_n^p}.$$

The proof of the second part is straightforward. \square

It follows directly from (9), that $\Gamma(s)$ tends to infinity, when s tends to zero. The following lemma quantifies this phenomenon. Although the statement seems to be well known, we were not able to find a reference and we therefore provide at least a sketch of the proof.

Lemma 15. Let $C \simeq 0.577 \dots$ denote the Euler constant. Then

$$\lim_{n \rightarrow \infty} \left(\frac{\Gamma(1/n)}{n} \right)^n = e^{-C}.$$

Proof. It is enough to show, that

$$\lim_{n \rightarrow \infty} n \cdot \log(\Gamma(1 + 1/n)) = -C,$$

which (by using the l'Hospital rule) follows from

$$\lim_{n \rightarrow \infty} \frac{\int_0^\infty s^{1/n} e^{-s} \log s \, ds}{\int_0^\infty s^{1/n} e^{-s} \, ds} = -C.$$

But the numerator of this fraction is equal to $\Gamma'(1 + 1/n)$ and its denominator to $\Gamma(1 + 1/n)$. The whole fraction is therefore equal to $\Psi(1 + 1/n)$ and $\Psi(1 + 1/n) \rightarrow \Psi(1) = -C$ as n tends to infinity, cf. [1, Section 6.3.2, p. 258]. \square

Next theorem shows, that if $\beta = p/n - 1$, then the measure $\theta_{p,\beta}$ promotes sparsity and one may even consider limiting behavior of n growing to infinity.

Theorem 16. Let $0 < p < \infty$ and let $n \geq 2$ and $1 \leq m \leq n$ be integers. Then

$$\sigma_{m-1}^{p,\infty}(\theta_{p,p/n-1}) \geq C_p^1 \cdot \frac{\Gamma(n+1)}{\Gamma(n-m+1)} \cdot \frac{\Gamma(n/p + n - m + 1)}{\Gamma(n/p + n + 1)}, \quad (35)$$

and

$$\sigma_{m-1}^{p,\infty}(\theta_{p,p/n-1}) \leq C_p^2 \cdot \frac{\Gamma(n+1)}{\Gamma(n-m+1)} \left\{ \frac{\Gamma(n/p+n-m+1)}{\Gamma(n/p+n+1)} + \frac{1}{m!} \cdot \left(\frac{e^{-1}}{\Gamma(1/n)} \right)^m \right\} \quad (36)$$

where C_p^1 and C_p^2 are positive real numbers depending only on p .

Furthermore, for every fixed $m \in \mathbb{N}$,

$$\frac{C_p^1}{\left(\frac{1}{p} + 1\right)^m} \leq \liminf_{n \rightarrow \infty} \sigma_{m-1}^{p,\infty}(\theta_{p,p/n-1}) \leq \limsup_{n \rightarrow \infty} \sigma_{m-1}^{p,\infty}(\theta_{p,p/n-1}) \leq \frac{C_p^2}{\left(\frac{1}{p} + 1\right)^m}, \quad (37)$$

where C_p^1 and C_p^2 are positive real numbers depending only on p .

Proof. First observe, that $n(\beta+1)/p = 1$ for $\beta = p/n - 1$ and therefore

$$\frac{\Gamma(n(\beta+1)/p)}{\Gamma(n(\beta+1)/p + 1/p)} = \frac{1}{\Gamma(1 + 1/p)}$$

depends only on p . Due to Lemma 14, we have to estimate

$$\mathbb{E} x_m^* \left(\prod_{i=1}^n x_i^{p/n-1} \right) = c_p^n \int_{\mathbb{R}_+^d} x_m^* \prod_{i=1}^n x_i^{p/n-1} e^{-x_1^p - \dots - x_n^p} dx. \quad (38)$$

Let $t = x_m^*$ and let us assume, that there is only one coordinate $j = 1, \dots, n$, such that $x_j = t$. Obviously, this assumption holds almost everywhere. Of course, we have n possibilities for j . Furthermore, $m-1$ from the remaining $n-1$ components of x are bigger than t and the remaining $n-m$ components are smaller. This allows to rewrite (38) as

$$\begin{aligned} & c_p^n n \binom{n-1}{m-1} \int_0^\infty t^{p/n} e^{-t^p} \left(\int_0^t u^{p/n-1} e^{-u^p} du \right)^{n-m} \times \\ & \quad \times \left(\int_t^\infty u^{p/n-1} e^{-u^p} du \right)^{m-1} dt \\ &= \frac{c_p^n n}{p^n} \binom{n-1}{m-1} \int_0^\infty \omega^{1/p+1/n-1} e^{-\omega} \left(\int_0^\omega s^{1/n-1} e^{-s} ds \right)^{n-m} \times \\ & \quad \times \left(\int_\omega^\infty s^{1/n-1} e^{-s} ds \right)^{m-1} d\omega. \end{aligned}$$

Let us denote

$$\gamma = \Gamma(1/n) = \int_0^\infty s^{1/n-1} e^{-s} ds \quad \text{and} \quad y(\omega) = \gamma^{-1} \cdot \int_0^\omega s^{1/n-1} e^{-s} ds. \quad (39)$$

Then $y(\omega)$ is a non-decreasing function of ω , $y(0) = 0$ and $\lim_{\omega \rightarrow \infty} y(\omega) = 1$. We denote by $\omega(y)$ its inverse function, i.e.

$$y = \gamma^{-1} \cdot \int_0^{\omega(y)} s^{1/n-1} e^{-s} ds, \quad 0 \leq y \leq 1. \quad (40)$$

Using this notation, we obtain

$$\mathbb{E} x_m^* \left(\prod_{i=1}^n x_i^{p/n-1} \right) = \frac{c_p^n \gamma^n}{p^n} n \binom{n-1}{m-1} \int_0^1 \omega(y)^{1/p} y^{n-m} (1-y)^{m-1} dy$$

and

$$\sigma_{m-1}^{p,\infty}(\theta_{p,p/n-1}) = \frac{\Gamma(n+1)}{\Gamma(m)\Gamma(n-m+1)} \int_0^1 \omega(y)^{1/p} y^{n-m} (1-y)^{m-1} dy, \quad (41)$$

where $\omega(y)$ is given by (40).

Step 1. Estimate from below

The estimate

$$\gamma y = \int_0^{\omega(y)} s^{1/n-1} e^{-s} ds \leq \int_0^{\omega(y)} s^{1/n-1} ds = n\omega(y)^{1/n}$$

implies together with Lemma 15

$$\omega(y) \geq \left(\frac{\gamma y}{n} \right)^n \geq c y^n$$

with c independent of n . This gives finally

$$\begin{aligned} \sigma_{m-1}^{p,\infty}(\theta_{p,p/n-1}) &\geq c^{1/p} \cdot \frac{\Gamma(n+1)}{\Gamma(m)\Gamma(n-m+1)} \cdot \int_0^1 y^{n/p+n-m} (1-y)^{m-1} dy \\ &= c^{1/p} \cdot \frac{\Gamma(n+1)}{\Gamma(m)\Gamma(n-m+1)} \cdot B(n/p+n-m+1, m) \\ &= c^{1/p} \cdot \frac{\Gamma(n+1)}{\Gamma(n-m+1)} \cdot \frac{\Gamma(n/p+n-m+1)}{\Gamma(n/p+n+1)}, \end{aligned}$$

where we used the Beta function (10) and the proof of (35) is complete.

Step 2. Estimate from above

Let us first take y , such that $1 - e^{-1}/\gamma \leq y \leq 1$. Then $-\ln(\gamma(1-y)) \geq 1$ and

$$\int_{-\ln(\gamma(1-y))}^{\infty} s^{1/n-1} e^{-s} ds \leq \int_{-\ln(\gamma(1-y))}^{\infty} e^{-s} ds = \gamma(1-y).$$

Hence,

$$\omega(y) \leq -\ln(\gamma(1-y)), \quad 1 - e^{-1}/\gamma \leq y \leq 1. \quad (42)$$

Finally, we observe, that

$$f : y \rightarrow \int_{Cy^n}^{\infty} s^{1/n-1} e^{-s} ds$$

is a convex function on \mathbb{R}_+ , $f(0) = \gamma$ and

$$\begin{aligned} f(1 - e^{-1}/\gamma) &= \int_{C(1-e^{-1}/\gamma)^n}^{\infty} s^{1/n-1} e^{-s} ds \\ &\leq \int_1^{\infty} s^{1/n-1} e^{-s} ds \leq e^{-1}, \end{aligned}$$

if we choose C so large, that $C(1 - e^{-1}/\gamma)^n \geq 1$ for all $n \in \mathbb{N}$. This is indeed possible, while a byproduct of Lemma 15 is also a relation $\lim_{n \rightarrow \infty} \gamma/n = 1$. Using the convexity of f , we obtain

$$f(y) \leq \gamma(1 - y), \quad 0 \leq y \leq 1 - e^{-1}/\gamma,$$

which further leads to

$$\omega(y) \leq Cy^n, \quad 0 \leq y \leq 1 - e^{-1}/\gamma. \quad (43)$$

We insert (42) and (43) into (41) and obtain

$$\sigma_{m-1}^{p,\infty}(\theta_{p,p/n-1}) \leq \frac{\Gamma(n+1)}{\Gamma(m)\Gamma(n-m+1)} \left\{ C^{1/p} I_1 + I_2 \right\}, \quad (44)$$

where

$$I_1 := \int_0^{1-e^{-1}/\gamma} y^{n/p+n-m} (1-y)^{m-1} dy$$

and

$$I_2 := \int_{1-e^{-1}/\gamma}^1 |\ln(\gamma(1-y))|^{1/p} y^{n-m} (1-y)^{m-1} dy.$$

The first integral may be estimated again using the Beta function, which gives

$$I_1 \leq B(n/p + n - m + 1, m). \quad (45)$$

We denote by k the uniquely defined integer, such that $1/p \leq k < 1/p + 1$ holds, and estimate

$$I_2 \leq \int_{1-e^{-1}/\gamma}^1 |\ln(\gamma(1-y))|^{1/p} (1-y)^{m-1} dy \leq I_{k,m} := \int_0^{e^{-1}/\gamma} |\ln(\gamma y)|^k y^{m-1} dy.$$

Next, we use partial integration to estimate $I_{k,m}$. We obtain

$$I_{k,m} = \frac{1}{m} \left(\frac{e^{-1}}{\gamma} \right)^m + \frac{k}{m} \cdot I_{k-1,m}.$$

Together with $I_{0,m} = 1/m \cdot (e^{-1}/\gamma)^m$, this leads finally to

$$I_{k,m} \leq \frac{(k+1)!}{m} \left(\frac{e^{-1}}{\gamma} \right)^m.$$

This, together with (44) and (45) finishes the proof of (36).

The proof of (37) then follows directly by Stirling's formula (11). \square

Remark 5. (i) Let us take $m = 0$. Then the formula (37) describes an essentially different behavior compared to the normalized cone and surface measure. Namely, the expected value of the largest coordinate of $x \in \Delta_p^n$ with respect to $\theta_{p,p/n-1}$ does not decay to zero with n growing to infinity. We shall demonstrate this effect also numerically in next section.

- (ii) If $m > 0$, then (37) shows, that $\sigma_m^{p,\infty}(\theta_{p,p/n-1})$ decays exponentially fast with m , as soon as n is large enough. That means, that for n large enough, the average vector of Δ_p^n exhibits a strong sparsity-like structure. Namely, its m -th largest component decays exponentially with m .
- (iii) We have chosen in (32) a different β for each n , namely $\beta_n = p/n - 1 > -1$. This was of course a crucial ingredient in the proof of Theorem 16. It is not difficult to modify the analysis of the proof of Theorem 16 to the situation, when $\beta > -1$ is fixed for all $n \in \mathbb{N}$. In this case we obtain again, that (up to logarithmic factors) $\sigma_0^{p,\infty}(\theta_{p,\beta})$ is equivalent to $n^{-1/p}$ with constants of equivalence depending on $p > 0$ and $\beta > -1$.
- (iv) Last, but not least, we observe, that one may choose $p = 1$ or even $p = 2$ in Theorem 16 and still obtains the exponential decay of coordinates as described by (37). It seems, that there is no significant connection between sparsity of an average vector of $x \in \Delta_p^n$ and the size of $p > 0$.

5 Numerical experiments

5.1 Cone measure

We would like to demonstrate the most significant effects of the theory also by numerical experiments. We start with the case of the cone measure. The key role is played by (14). It may be interpreted in the following way. To generate a random point on Δ_p^n with respect to the normalized cone measure, it is enough to generate $\omega_1, \dots, \omega_n$ with respect to the density $c_p e^{-t^p}$, $t > 0$ and then calculate

$$\frac{(\omega_1, \dots, \omega_n)}{(\sum_{j=1}^n \omega_j^p)^{1/p}} \in \Delta_p^n.$$

This method is very practical, as the running time of this algorithm depends only linearly on n .

Let us note, that the values of ω_i may be generated very easily. For example the package *GNU Scientific Library* [26] implements a random number generator with respect to the gamma distribution using the method described in the classical work of Knuth [31]. Using this package, we generated 10^8 random points $x \in \Delta_p^n$ for $n = 100$ and $p \in \{1/2, 1, 2\}$ to approximate numerically the value of $n^{1/p} \cdot \int_{\Delta_p^n} x_m^* d\mu_p(x)$. The result may be found in the Figure 1.

5.2 Tensor measures

As pointed out in Remark 4, point (iii), a random point on Δ_p^n with respect to $\theta_{p,\beta}$ may be generated in the following way. We generate $\omega'_1, \dots, \omega'_n$ with respect to the density $c_{p,\beta} t^\beta e^{-t^p}$, $t > 0$, where $c_{p,\beta}^{-1} = \int_0^\infty t^\beta e^{-t^p} dt$ is a normalizing constant and we consider the vector

$$\frac{(\omega'_1, \dots, \omega'_n)}{(\sum_{j=1}^n (\omega'_j)^p)^{1/p}} \in \Delta_p^n.$$

Also this may be easily done with the help of [26]. We generated again 10^8 random points $x \in \Delta_p^n$ with respect to $\theta_{p,p/n-1}$ for $n = 100$ and $p \in \{1/2, 1, 2\}$. Then we used those points to numerically approximate the expression $\log_{10}(\int_{\Delta_p^n} x_m^* d\theta_{p,p/n-1})$.

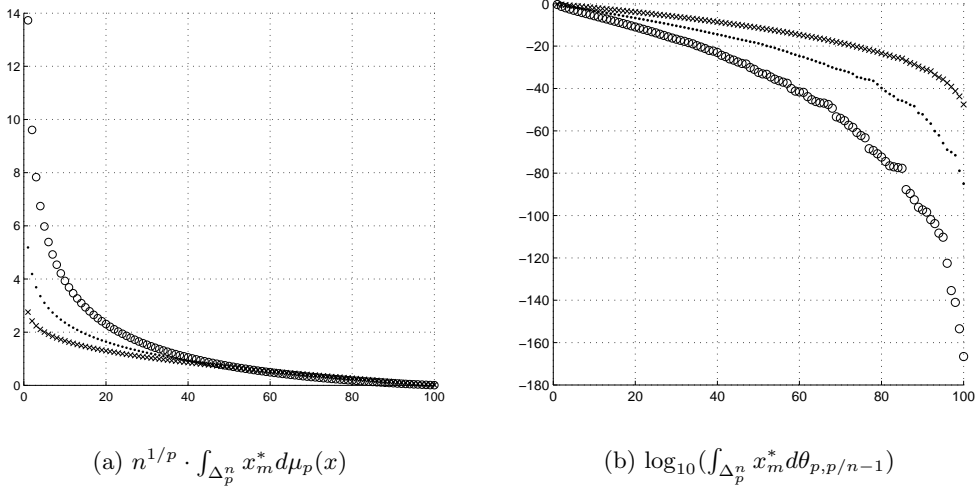


Figure 1: Approximations of $n^{1/p} \cdot \int_{\Delta_p^n} x_m^* d\mu_p(x)$ (left) and $\log_{10}(\int_{\Delta_p^n} x_m^* d\theta_{p,p/n-1})$ (right) for $n = 100$, $p = 1/2(\circ)$, $p = 1(\bullet)$ and $p = 2(\times)$ based on sampling of 10^8 random points.

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